Review on Regular Black Holes in General Relativity

An Undergrad View Through General Relativity

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Summary

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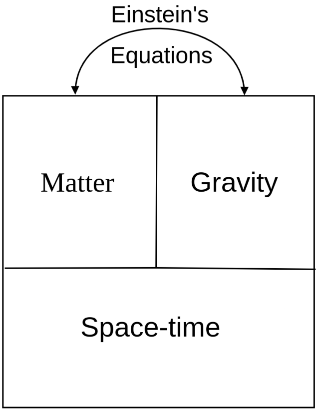
Introduction

- Special Relativity
- The Big Picture:



Fonte: Caio César

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Introduction

- "Por agir do mesmo modo em todas as formas de energia, a gravidade deve se manifestar como um fenômeno geométrico"(Gonzalo Olmo).
- This underlies that notion in order to talk about gravity, the notion of **spacetime**

Definition (Spacetime)

Spacetime is a four-dimensional topological manifold with a smooth atlas carrying a torsion-free connection compatible with a Lorentzian metric and a time orientation satisfying the Einstein's equations.

- Although mathematical rigorous, such a definition is necessary in order to talk about gravity, though it won't be needed throughout this presentation.

General Relativity

- As seen before, special relativity and Minkowski **flat** spacetime is not general.
- We want to pursue a description of spacetime in which it locally looks like Minkowski, but over extended regions, has non-trivial curvature.
- By adding non-inertial reference frames + differential geometry:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1}$$

General Relativity

- We'll restrict ourselves of coordinates like $x^\mu = \{t, r, \theta, \varphi\}$, and diagonal metrics.
- For calculating distances in spacetime:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

- With the line element in mind, trajectories are divided into three categories:
  - $ds^2 < 0 \to$ time-like trajectories ($\mathrm{m} \neq 0$ particles)
  - $ds^2 = 0 \to$ light-like trajectories (Photon)
  - $ds^2 > 0 \to$ space-like trajectories (Tachyon)
- Tangent velocity:

$$\|u^\mu\|^2 = g_{\mu\nu} u^\mu u^\nu \tag{3}$$

  - Why can't a time-like trajectory become space-like trajectory?
- Furthermore, in spherical coordinates, the line element:

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2 \tag{4}$$

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Black Holes

If there should really exist in nature any bodies, whose density is not less than that of the sun, and whose diameters are more than 500 times the diameter of the sun, since their light could not arrive at us; or if there should exist any other bodies of a somewhat smaller size, which are not naturally luminous; of the existence of bodies under either of these circumstances, we could have no information from sight; yet, if any other luminous bodies should happen to revolve about them we might still perhaps from the motions of these revolving bodies infer the existence of the central ones with some degree of probability, as this might afford a clue to some of the apparent irregularities of the revolving bodies, which would not be easily explicable on any other hypothesis; but as the consequences of such a supposition are very obvious, and the consideration of them somewhat beside my present purpose, I shall not prosecute them any further..
— John Michell, 1784

Black Holes

- What exactly is a Black Hole?

Definition (Black Hole)

A Black Hole is a region of spacetime in which there is no escape, such that any object or light ray that enters region II will never escape from this region and will end its existence in $r = 0$, such that if $\mathcal{M} \not\subseteq J^{-}(\mathscr{I}^{+})$, a strongly asymptotically spacetime allows the existence of such region.

$$\mathcal{B} = \mathcal{M} - J^{-}(\mathscr{I}^{+})$$

- In which $\mathcal{M}$ is a manifold associated with a metric g_{ab}, so that, together, $(\mathcal{M}, g_{ab})$ forms a asymptotically flat spacetime.
- $J^{-}(\mathscr{I}^{+})$ is called the causal step of the infinite future light-like.

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Black Holes

- The correspondence region of a black hole does not include any trajectories that are time-like or light-like, such that they cannot influence $\mathcal{B}$ even in a infinity amount of time.
- This means that the regions of infinity future time-like and light-like has no causality connection with the BN itself, implying a limitation by a boundary.

Definition (Event Horizon)

The event horizon is defined as the the light-like surface in the boundary of $\mathcal{B}$, such that

$$\mathcal{H} = \mathcal{M} \cap j^{-}(\mathscr{I}^{+})$$

- $j^{-}(\mathscr{I}^{+})$ is the boundary of $J^{-}(\mathscr{I}^{+})$
- This definition implies that for any particle or photon, once reached $\mathcal{H}$, they can never reach infinity anymore.

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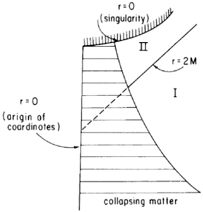
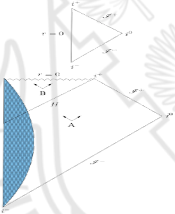
Black Holes

- So the internal region of a black hole has no causality relation with the exterior, i.e, all events restricted to $r = 2M$, have no effect on the outside.



Fonte: Fig 6.11 - Robert M. Wald, General Relativity



Fonte: Juliano Neves

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Schwarzschild Solution

- All of this takes to talk about the first black hole solution, the Schwarzschild solution:
  - Steady state space-times.
  - Spherical symmetry.

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2); \quad f(r) = 1 - \frac{2m}{r} \quad (5)$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (6)$$

- $\lim_{r \to 0} f(r) = -\infty \Longrightarrow$ Divergence of (6).
- $\lim_{r \to 2m} f(r) = 0 \Longrightarrow$ Event Horizon limit.

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Reissner-Nordström Solution

- Same conditions as Schwarzschild + charge, i.e, up to a E&M field.

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{7}$$

- $f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{4\pi\epsilon_0 r^2}$
- $\lim_{r\to 0} f(r) = \pm\infty$

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Regular Solutions

- So there is a natural singularity embedded in some BN solutions, however, the black definition doesn't take into account singularities, so regular solutions must somehow exist.
- De Sitter Solution:

$$ds^2 = -\left(1 - \frac{\Lambda}{3}\right)dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3}} + r^2(d\theta^2 + \sin^2\theta d^2\varphi) \tag{8}$$

  - Spherical Symmetry + cosmological constant
- $\Lambda > 0 \rightarrow$ De Sitter spacetime.
- $\Lambda < 0 \rightarrow$ Anti-de Sitter spacetime (AdS-CFT correspondence).

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Regular Solutions

- If we take a look back at the Schwarzschild solution, the mass if constant, but if $m = m(r)$ and r goes to zero:

$$\therefore m(r) = \frac{Mr^3}{(r^2 + e^2)^{\frac{3}{2}}} \implies f(r) \approx 1 - \frac{2M}{e^3} r^2 \tag{9}$$

- Substituting $f(r)$ into equation (5), a De-Sitter metric is obtained for small values of r
- Bardeen's effective mass + (5): describes a spherical black hole with a nucleus likewise De-Sitter, making $r = 0$, not singular in $\mathcal{M}$ anymore.

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A Few Remarks

- In order to clearly see the regularity:

$$\lim_{r\to 0} R^{\mu\nu\alpha\rho}R_{\mu\nu\alpha\rho} = 96\left(\frac{M}{e^3}\right)^2 \not\approx \frac{1}{r} \tag{10}$$

- Therefore, Bardeen's black hole shows up as a regular solution
- The source(e), in Bardeen's solution, according to Ayon-Beato & Garcia, is taken as a magnetic monopole, from a non-linear electrodynamics expressed by the action:

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi} - \frac{1}{4\pi}\mathcal{L}(F)\right) \tag{11}$$

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Thanks for listening

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