The Old Quantum Mechanics

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"Science is built up of facts, as a house is with stones. But a collection of facts is no more a science than a heap of stones is a house.".

Henri Poincare

Contents

Chapter 1

From Continuous to Discrete

Towards the end of the 19Th century physics was flourishing. The physics of thermodynamics was at one of its highest points, its principles had the key roll on the kick start of the industrial revolution. Faraday had recently discovered how to generate electricity. Furthermore, James Clark Maxwell grand unification of electricity and magnetism has let to the remarkable realization that light is a electromagnetic wave.

With the power of these theories in hand, physicists of the time had began using these then, to address some of the biggest, and thought to the the only since physicists of the day thought that after solving this problem, physics would be just about getting more an more accurate value for some constants, like the Boltzmann constant. **How hot objects emitted light?**

1.1 Black Body Radiation

Physicists wanted to explain how the brightness, i.e, intensity of the emitted radiation related to the wavelength, at a fundamental level. But this act would require to understand how matter interacted with radiation.

1.1.1 What is a black body?

Dark objects tend to absorb more of the light incident upon them and reflect less than lighter objects, which is why they tend to heat up more quickly. So a object that absorbs all of the light incident upon it, is referred by physicists as a black body.

A very good approximation of a black body is the so called Jeans Cube, named after James Jeans$(1877 - 1946)$ who along with Lord John Rayleigh$(1842 - 1919)$ provided one of the first attempts to explain such phenomena. The *Jeans Cube* is a metallic box with a small hole at one of the sides. Radiation will naturally come in from the outside and reflect inside it back and forth until total absorption, but only a very tiny amount of radiation will come out of the cube, so from the hole perspective all radiation is absorbed, thus making it act as a black body.

1.1.2 Experimental Results

Experiments of the time made it possible to define a quantity called **spectral radiance**.

$$I(T) = \int_0^\infty d\lambda R(\lambda) \tag{1.1.1}$$

From these experiments, patterns in the data were revelled in such a way that two empirical laws emerged:

(i) *Stephan-Boltzmann Law*: The brightness of a object is proportional to the forth power of the temperature.

$$I(T) = \sigma T^4 \tag{1.1.2}$$

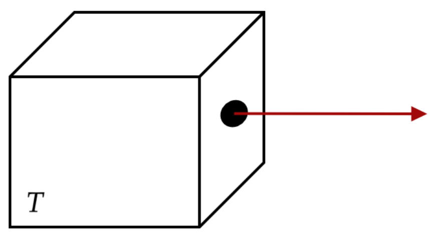
(ii) *Wien's Displacement Law*: The temperature of a object is inversely proportional to the the peak of wavelengths.

$$T\lambda_{max} = \text{constant} \tag{1.1.3}$$

So the first question asked from these experimental results was: Is it possible to derive $R(\lambda)$ from first Principles?

1.1.3 The Rayleigh-Jeans Model

So Lord Rayleigh and James Jeans realized it would be more useful and simpler if they specified the spectrum of radiation inside the Jeans Cube in terms of the energy density, rather than spectral radiance.



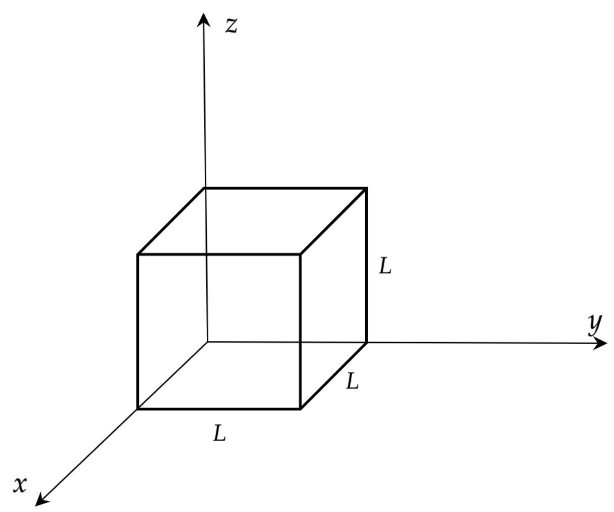
The energy density is simply the energy contained in unit volume of the cube at given temperature T, such that $\rho = \rho(\nu)\ \forall \nu \in [\nu, \nu + d\nu]$. Since $\rho(\nu) \propto R(\nu)$ then one can be calculate from the other. With that in hands, a three step strategy was made.

1. Count the number of waves that can "fit" inside the cube.

2. Calculate the average energy of these waves in thermal equilibrium.

3. Combine the last two results to then find the energy density.

Therefore, first making a concrete analysis of the stationary waves inside the cube.



The electromagnetic waves hitting on the wall will have three components. thus all radiation along a component is reflected upon the wall, combining the incident and reflected waves, therefore forming a stationary wave along some direction, say the x-direction. This directly implies that electric field is parallel to the wall. But a metallic wall can not support a electric field parallel to the surface electric charges can only flow in a way that $E\Big|_{wall} = 0$, then $E(0) = 0 = E(L)$. Which clearly is also valid for the other directions.

However, the net electric field is described by,

$$E_{net} = 2A\sin(kx)\cos(\omega t) \tag{1.1.4}$$

By making use of boundary conditions, at $x = 0$,

$$2A\sin(kx)\cos(\omega t) = 0 = E_{net}(x,t) \tag{1.1.5}$$

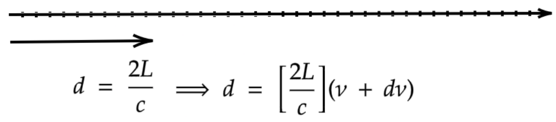
And at $x = L$,

$$2A\sin(kx)\cos(\omega t) = 0 \iff \sin(kL) = 0 \tag{1.1.6}$$

$$\iff kL = n\pi \iff k = \frac{n\pi}{L} \iff \lambda = \frac{2L}{n}\ \forall n \in \mathbb{Z}$$

$$\implies \nu = \frac{c}{\lambda} \iff \nu = \frac{nc}{2L}$$

So having found a expression to the frequency, what is the frequency density? Representing these allowed values of frequencies in terms of a diagram consisted of integer values lying on a line in which we can specify the distance from the origin.



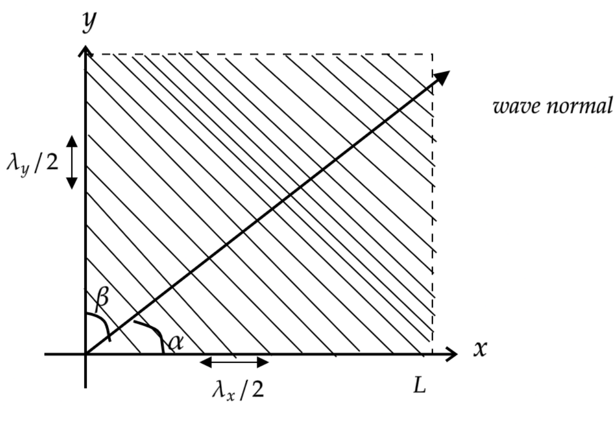
Notice how the number of points falling between the two limits do not depend on ν but on $d\nu$. Which can be clearly seen by calculating and defining the number of states on the internal, i.e, $\forall\nu \in [\nu, \nu + d\nu]$.

$$N(\nu)d\nu = \frac{2L}{c}(\nu + d\nu) - \frac{2L}{c}\nu \iff N(\nu) = d\nu\frac{2L}{c}d\nu$$

But for each of the allowed frequencies there are actually two independent waves corresponding to the possible states of polarization.

$$\therefore N(\nu)d\nu = \frac{4L}{c}d\nu \tag{1.1.7}$$

This provides the initial framework on how to calculate the number of allowed standing waves inside the Jeans Cube. However, so far it was only considered one spatial dimension, thus it necessary to expand to three. But for purposes of simpler drawings, in the following it will be considered two dimensions which can easily be generalized to three dimensions.



Considering radiation of wavelength λ propagating in a direction defined by the two angles α, β. The radiation formed in that direction must form a stationary wave. The diagram indicates the location of some of the fixed nodes of the standing wave by a set of lines perpendicular to the propagation direction. The distance between these nodal lines is half a wavelength as is true for all stationary wave patterns. By the geometry of the diagram,

$$\lambda_x = \frac{\lambda}{\cos\alpha} \tag{1.1.8}$$

$$\lambda_y = \frac{\lambda}{\sin\beta} \tag{1.1.9}$$

$$\Longrightarrow n_x = \frac{2L\cos\alpha}{\lambda};\ n_y = \frac{2L\cos\beta}{\lambda} \tag{1.1.10}$$

then, naturally for three dimensions,

$$\lambda_z = \frac{2L}{nz} \iff n_z = \frac{2L\cos\gamma}{\lambda} \tag{1.1.11}$$

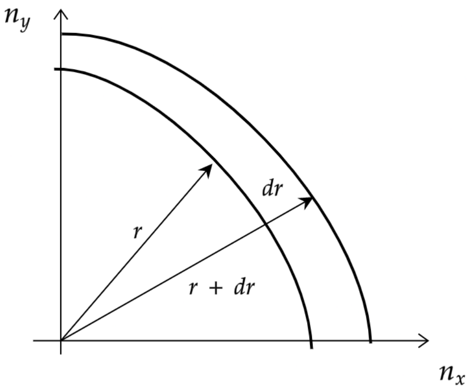
Considering then,

$$n_x^2 + n_y^2 + n_z^2 = \frac{4L^2}{\lambda^2}[\cos^2\alpha + \cos^2\beta + \cos^2\gamma] \iff \lambda = \frac{2L}{(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)^{\frac{1}{2}}}$$

expressing it in terms of frequencies,

$$\nu = \frac{c}{2L}[n_x^2 + n_y^2 + n_z^2]^{\frac{1}{2}} \tag{1.1.12}$$

Now the question comes to the relationship of this expression to determine the density of states in the three dimensional cube. If for instance, it is assumed that a sphere has a radius r, for questions of simplification of calculations, in which the three axis represent each n_i; $i = 1, 2, 3$, with center at the origin, then, it is possible to determine the density of states that lie in the interval $[r, r + dr]$. Considering just a section of the sphere,



$$r = (n_x^2 + n_y^2 + n_z^3)^{\frac{1}{2}}$$

$$\nu = \frac{c}{2L}(n_x^2 + n_y^2 + n_z^3)^{\frac{1}{2}}$$

$$\nu = \frac{cr}{2L} \iff r = \frac{2L\nu}{c} \tag{1.1.13}$$

$$\therefore \frac{\mathrm{d}r}{\mathrm{d}\nu} = \frac{2L}{c} \iff \mathrm{d}r = \frac{2L}{c}\mathrm{d}\nu \tag{1.1.14}$$

It can be saw from the diagram that the number of states within the radio will be equal to the volume of the small segment labeled $\mathrm{d}r$, which for the whole sphere would be equal to the surface area of the sphere itself multiplied by a factor $\mathrm{d}r$. However, since that is just about $\frac{1}{8}$ of the sphere's volume, the expression is divided by it.

$$N(r)\mathrm{d}r = \frac{1}{8}4\pi r^2\mathrm{d}r \iff N(r)\mathrm{d}r = \frac{\pi r^2}{2}\mathrm{d}r \tag{1.1.15}$$

By equations (1.1.13) and (1.1.14) rewriting in terms of frequency,

$$N(\nu)\mathrm{d}\nu = \frac{\pi}{2}\Big(\frac{2L}{c}\Big)^3\nu^2\mathrm{d}\nu \iff N(\nu)\mathrm{d}\nu = \frac{4\pi L^3}{c^3}\nu^2\mathrm{d}\nu$$

However, the two possible polarization states need to be accounted for,

$$\therefore N(\nu)\mathrm{d}\nu = \frac{8\pi L^3\nu^2}{c^3}\mathrm{d}\nu \tag{1.1.16}$$

The expression above for the number of states as a function of the frequency allows to count the number of allowed wave frequencies inside the cube. Now it is needed to calculate the average energy of each of these waves.

In order to calculate the energy of each wave, Rayleigh-Jeans has to rely on the well established result from the classical thermodynamics, the Equipartition Theorem. According to it, if there is a gas at thermal equilibrium comprising billions and billions of small particles in random motion, then, the available thermal energy within the system will be on average equally distributed between the particles of the gas. If for instance one considers a box containing some sort of monotonic gas, then he kinetic energy of each atom can be split into three modes corresponding to each direction. So if X joules were added to the system, $\frac{X}{3}$ were to be equally added to each direction. For a system in thermal equilibrium each mode(i.e, degree of freedom) has an average energy equal to,

$$\langle KE\rangle = \frac{1}{2}kT$$

So for 3 degrees of freedom, it would have an average energy of $\langle E\rangle = \frac{3}{2}kT$ and the total energy would be equal to the average.

By this, Rayleigh and Jeans reasoned that the same statistical law must hold for the waves inside the cube. This was because the law of equipartition of energy was believed to apply to any classical system in equilibrium.

For the stationary waves at hand, each with one degree of freedom that corresponds to the eletric field amplitudes. But, each sinusoidal oscillating standing wave has a total energy that is twice its average, therefore

$$E = kT \tag{1.1.17}$$

This have to do with a much more deep understanding of thermodynamics, that is about the Boltzmann distribution, which provides complete information about the energy of the entities in the system, including the average energy.

$$\therefore \langle E\rangle = \frac{\displaystyle\int_0^\infty \mathrm{d}E\ Ee^{-E/kT}}{\displaystyle\int_0^\infty \mathrm{d}E\ e^{-E/kT}} \tag{1.1.18}$$

Solving the integral on the denominator,

$$\int_0^\infty \mathrm{d}E\ e^{-E/kT};\ kTu = -E \Longrightarrow -kT\mathrm{d}u = \mathrm{d}E$$

$$\Longrightarrow \int_0^\infty \mathrm{d}u\ (-kTe^u) = -kT\ e^u\Big|_0^\infty = kT$$

$$\int_0^\infty \mathrm{d}E\ e^{-E/kT} = kT \tag{1.1.19}$$

Now performing the integral on the numerator. First taking the derivative with respect to the energy of the argument,

$$\therefore \frac{\mathrm{d}(Ee^{-E/kT})}{\mathrm{d}E} = e^{-E/kT} - \frac{E}{kT}e^{-E/kT}$$

$$\Longrightarrow \int_0^\infty \mathrm{d}E\ \frac{\mathrm{d}(Ee^{-E/kT})}{\mathrm{d}E} = \int_0^\infty \mathrm{d}E\ e^{-E/kT} - \frac{1}{KT}\int_0^\infty \mathrm{d}E\ Ee^{-E/kT}$$

$$\Longleftrightarrow \int_0^\infty \mathrm{d}E\ Ee^{-E/kT} = kT\int_0^\infty \mathrm{d}E\ e^{-E/kT} - \int_0^\infty \mathrm{d}E\ \frac{\mathrm{d}(Ee^{-E/kT})}{\mathrm{d}E}$$

$$\Longleftrightarrow \int_0^\infty \mathrm{d}E\ Ee^{-E/kT} = (kT)^2 \tag{1.1.20}$$

Therefore the average energy that comes from the Boltzmann distribution is given by (1.1.17), as already said earlier.

Calculating now the energy per volume in a interval range $[\nu, \nu+\mathrm{d}\nu]$, by equations (1.1.16) e (1.1.17)

$$\rho(\nu)\mathrm{d}\nu = \frac{8\pi L^3}{c^3}\nu^2\frac{kT}{L^3}\mathrm{d}\nu$$

$$\therefore \rho(\nu)\mathrm{d}\nu = \frac{8\pi\nu^2}{c^3}kT\mathrm{d}\nu \tag{1.1.21}$$

The equation above is the energy density as a function of the frequency, but what about a energy density as a function of wavelength.

Since,

$$\rho(\nu)\mathrm{d}\nu = -\rho(\lambda)\mathrm{d}\lambda \tag{1.1.22}$$

and by expression that relates frequency and wavelength.

$$\rho(\lambda)\mathrm{d}\lambda = -\frac{8\pi\nu^2 kT}{c^3}\mathrm{d}\nu$$

$$\rho(\lambda)\mathrm{d}\lambda = \frac{8\pi kT}{\lambda^4}\mathrm{d}\lambda \tag{1.1.23}$$

Equation (1.1.23) is the reason on to why the Rayleigh-Jeans model is called the *The Ultraviolet Catastrophe*. Notice how for small numbers of wavelength, the energy density simply explodes rapidly, implying in a infinity amount of energy in the system, which would violate the conversation of energy.

On the other hand, for large numbers of wavelength, the model would perfect agree with experiments, so there were something not that wrong that could be fixed to give a better prediction on the phenomena.

1.1.4 Max Plank Comes Into Play

Although the Rayleigh-Jeans model failed catastrophically for small wavelengths, for long wavelengths it matches the experiment with a incredible accuracy, which would suggest that in the limit of long wavelengths, the average energy would indeed be such that it equals kT.

From experimental data, Plank concluded that the energy should at any cost depend on the frequency, since at low wavelengths it should tend to zero. The key to this would be making adjustments to the Boltzmann distribution calculation.

$$\therefore \langle E\rangle = \frac{\displaystyle\int_0^\infty \mathrm{d}E\ Ee^{-E/kT}}{\displaystyle\int_0^\infty \mathrm{d}E\ e^{-E/kT}} \longrightarrow \langle E\rangle = \frac{\displaystyle\sum_{n=0}^{\infty} n\varepsilon \exp\Big(-\frac{n\varepsilon}{kT}\Big)}{\displaystyle\sum_{n=0}^{\infty} \exp\Big(-\frac{n\varepsilon}{kT}\Big)} \tag{1.1.24}$$

Notice that the denominator is quite similar to a geometric sum of the form

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \longrightarrow \sum_{n=0}^{\infty} \exp\Big(-\frac{n\varepsilon}{kT}\Big) = \frac{1}{1-\exp\Big(-\dfrac{\varepsilon}{kT}\Big)}$$

But the numerator is not that simple and requires much more thought about it.

$$\varepsilon\sum_{n=0}^{\infty} n\exp\Big(-\frac{n\varepsilon}{kT}\Big) = -kT\varepsilon\sum_{n=0}^{\infty} n - \frac{n}{kT}\exp\Big(-\frac{n\varepsilon}{kT}\Big) = -kT\varepsilon\frac{\mathrm{d}f}{\mathrm{d}\varepsilon}$$

such that $f(\varepsilon) = \displaystyle\sum_{n=0}^{\infty} \exp\Big(-\frac{n\varepsilon}{kT}\Big) = \frac{1}{1-\exp\Big(-\dfrac{\varepsilon}{kT}\Big)}$

$$\therefore \frac{\mathrm{d}f}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Bigg(\frac{1}{1-\exp\Big(-\dfrac{\varepsilon}{kT}\Big)}\Bigg) = -\frac{1}{kT}\frac{e^{\varepsilon/kT}}{(1-e^{\varepsilon/kT})^2}$$

Getting to the the new relation,

$$\langle E\rangle = \frac{\displaystyle\sum_{n=0}^{\infty} n\varepsilon \exp\Big(-\frac{n\varepsilon}{kT}\Big)}{\displaystyle\sum_{n=0}^{\infty} \exp\Big(-\frac{n\varepsilon}{kT}\Big)} = \frac{\varepsilon}{e^{\varepsilon/kT}-1} \tag{1.1.25}$$

Notice then from the above equation,

$$\begin{cases} \varepsilon <<< kT \longrightarrow e^{\varepsilon/kT} \approx 1 + \dfrac{\varepsilon}{kT} \longrightarrow \langle E\rangle \approx kT \\ \\ \varepsilon \to \infty \longrightarrow e^{\varepsilon/kT} \to \infty \longrightarrow \langle E\rangle \to 0 \end{cases}$$

This relation from the limits would directly simply that the energy ε would depend on the frequency such that

$$\varepsilon \propto \nu \iff \varepsilon = h\nu \tag{1.1.26}$$

h would be the constant of proportionally, the Plank constant, that later would be put in a position of one of the nature's fundamental constants.

Now the average energy is written as,

$$\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1} \tag{1.1.27}$$

Putting it together with equation (1.1.16), the new energy density is given by,

$$\rho(\nu)\mathrm{d}\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT} - 1}\mathrm{d}\nu \tag{1.1.28}$$

$$\rho(\lambda)\mathrm{d}\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}\mathrm{d}\lambda \tag{1.1.29}$$

Which would perfectly match the experimental data and make a great prediction on the phenomena of black body radiation.

1.2 Almost a Good Model: The Bohr Model

It was largely thought that atoms could not be directly observed and therefore, one could not make valid conclusions about their existence. But there were also those who supported the idea of atoms, referring to the remarkable success of the kinetic theory of gases from thermodynamics.

At the beginning of the 20th century, some experimental results came to prove they could only be explained by calling a atomic theory.

In 1997, J.J Thompson($1856-1940$) proved by experiment that tiny negatively charged particles called electrons, can be extracted from atoms, leaving behind positively charged ions. Which made crystal clear that atoms are not as the origin of its name, but rather complex systems(*literally*[1]) formed by positively and negatively parts.

1.2.1 Problems with the nuclear model

Although Rutherford's model was way better than Thompson's, for example. Then, if the electrons are believed to exist in the space surrounding the nucleus, then what are *exactly* they doing? They can't just be standing still because the electrostatic force of attraction between the nucleus and the electrons, would surely cause the electrons to be pulled into the nucleus. By this, all atoms are naturally unstable and therefore should collapse in a very short amount of time.

So the solution to this would be that likewise the earth orbiting the sun, the electrons are orbiting the positively charged nucleus, but that is a useless model as well. According to electromagnetism, surrounding every charged particle there exist a electric field, if a charged particle moves in a circular orbit, then the acceleration of the particle will cause a ripple in the surrounding electromagnetic field. Then, as the electron orbit the nucleus, it would radiate energy, and therefore as it loses energy it will slow down and spiral into the nucleus.

Furthermore, likewise Einstein and Plank before him, Bohr($1885-1962$) realized that attempting to resolve this problem with only classical physics was doomed to failure. Instead he took the the idea that the energy is transferred only by discreetly values, i.e, in chunks or quanta rather than in a continuous fashion as predicted by classical mechanics.

So Bohr hypothesized that in order for atoms to be stable there shall exist a stable configuration of the electron orbiting the nucleus. So he suggested that the transition to these states would cause the emission of light. Since this approach did not require anything to do with first principles from classical mechanics, there was needed the existence of postulates.

1.2.2 The Bohr Postulates

(i) The electron moves in a circular orbit centered on the nucleus

(ii) Despite the electron's acceleration, the electron does not continuously radiate energy, so its total energy remains constant.

[1] https://www.jlab.org/news/stories/now-presenting-visualization-proton

(iii) The magnitude of the electron's angular momentum is quantized in units of Plank's constant divided by 2π [2]

(iv) Electrons can move from higher to lower energy levels by emitting photons with energy equal to the difference between levels.

According to classical mechanics, the angular momentum of a orbiting electron is given by,

$$\mathbf{L} = m_e(\mathbf{v} \times \mathbf{r}) \iff \|\mathbf{L}\| = m_e \|\mathbf{v}\| \|\mathbf{r}\| \sin\theta$$

Since in the case of circular orbit, $\mathbf{v} \perp \mathbf{r}$

$$\|\mathbf{L}\| = m_e \|\mathbf{v}\| \|\mathbf{r}\| \tag{1.2.1}$$

Notice from the above expression that since velocity and radius can take any value, so the angular momentum would. But by the third Bohr postulate,

$$L = \frac{nh}{2\pi} \tag{1.2.2}$$

$$\iff \frac{nh}{2\pi} = m_e vr \iff r = \frac{nh}{2\pi m_e v} \tag{1.2.3}$$

But there still is a need to describe the motion of the electron and its orbits around the nucleus. Bohr assumed that the electron was healed in its orbit by the Coulomb's force of attraction.

$$F = \frac{e^2}{4\pi\epsilon_0 r^2} \iff F = \frac{m_e v^2}{r}$$

$$\iff \frac{e^2}{4\pi\epsilon_0 r^2} = \frac{m_e v^2}{r} \iff v^2 = \frac{e^2}{4\pi\epsilon_0 m_e r}$$

$$r^2 = \frac{n^2h^2}{4\pi^2 m_e^2 v^2} \implies r = \frac{n^2h^2\epsilon_0}{e^2 m_e \pi} \tag{1.2.4}$$

$$\implies v = \frac{e^2}{2\pi\epsilon_0 h} \tag{1.2.5}$$

By equation (1.2.4), it is clear that the electron is only allowed to orbit the nucleus at certain distances depending on the value of n.

$$n = 1 \implies r_1 = 0.53 \cdot 10^{-10}\text{m}$$

$$n = 2 \implies r_2 = 2.10 \cdot 10^{-10}\text{m}$$

$$n = 3 \implies r_3 = 4.80 \cdot 10^{-10}\text{m}$$

$$\cdots$$

Once Bohr had determined the radio of the allowed electron orbits, his next task was to determine the energy of each of theses orbits. In order to do this, he had to firstly think classically, through energy conservation.

$$E = T + V$$

[2] Bohr did not know at first that it was by units of Plank's constant, it was just made as Plank's constant because angular momentum and h have the same units.

For a potencial given by $V = -\dfrac{e^2}{4\pi\epsilon_0 r}$

$$E = \frac{1}{2}m_e v^2 - \frac{e^2}{4\pi\epsilon_0 r}$$

By equation (1.2.5),

$$E = \frac{m_e e^4}{8h^2\epsilon_0^2 n^2} - \frac{m_e e^4}{4h^2\epsilon_0^2 \pi n^2}$$

$$\iff E = -\frac{1}{n^2}\frac{m_e e^4}{8h^2\epsilon_0^2}$$

$$\therefore E = -\frac{2.18 \cdot 10^{-18}}{n^2}\text{J} \tag{1.2.6}$$

As one can see from equation (1.2.6), the orbitals energy are very small, so physicists often write the energy using different system of units called electronvoltz rather than Joules[3].

$$\therefore E_T = -\frac{13.6}{n^2}\text{ eV} \tag{1.2.7}$$

For the electron's first six orbits:

$$\begin{cases} E_1 = -13.6 \text{ eV} & E_4 = -0.85 \text{ eV} \\ E_2 = -3.4 \text{ eV} & E_5 = -0.54 \text{ eV} \\ E_3 = -1.3 \text{ eV} & E_6 = -0.38 \text{ eV} \end{cases}$$

Bohr than realized that in order for a electron in a ground state to move to a higher energy state it would need to absorb just the right amount of energy corresponding to the exact difference between two energy levels.

Imagine that a electron in a ground state is excited to the $n = 2$ level. When it dexcites and drop back-down to the ground state it would emit a photon.

$$E = h\nu = h\frac{c}{\lambda}$$

Since $E_2 \to E_1 \implies \Delta E = 13.6 - 3.4 = 10.2eV$ as the energy of the emitted photon. Also, by this, the wavelength of the emitted photon would be,

$$\lambda = \frac{c}{\Delta E} = 122 \text{ nm}$$

This type of radiation emitted is a ultraviolet region. So photons emitted from this transition are not directly visible. Likewise, electrons transitions from $n = 3, 4, 5, 6$ energy levels down to ground state would all cause the emission of ultraviolet photons. This series of wavelengths is known as the Lyman series.

So the outcome natural question to ask at this stage is whether any electron transitions correspond to the emission of visible light. Considering electron transitions from the $n = 3, 4, 5, 6$ to $n = 2$ energy levels.

[3] A electron-volt is defined as the energy transferred to a electron when its accelerated through a potential difference of 1V. Since $V = \frac{U}{q} \iff U = qV \;\therefore 1eV = 1 \cdot e = 1.6 \cdot 10^{-19}$ J

$$\begin{cases} E_3 \rightarrow E_2 \Longrightarrow 654 \text{ nm} \\ E_4 \rightarrow E_2 \Longrightarrow \lambda = 488 \text{ nm} \\ E_5 \rightarrow E_2 \Longrightarrow \lambda = 435 \text{ nm} \\ E_6 \rightarrow E_2 \Longrightarrow \lambda = 412 \text{ nm} \end{cases}$$

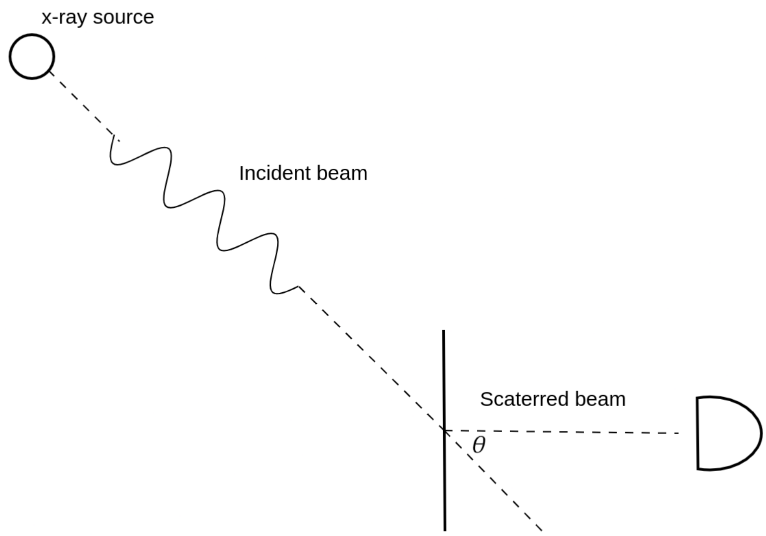
Which would be the electrons transitions that would eventually emit visible light.

1.3 Photons Exist: The Compton Scattering

The beginning of the 20th century represented a period of unprecedented progress in the field of fundamental physics. But much of this progress centered around a seemingly simple and innocuous question about the nature of light. What is light and how does it interact with matter?

By the electromagnetic theory of Maxwell, physicists were led to believe that light was a wave. However, by the black-body radiation(section 1.1) model of Max Plank and the photoelectric effect of Albert Einstein, there were born some evidence that it could be a particle as well.[4]

Compton experiment involved directing a beam of light, i.e, x-rays, of sharply defined wavelength onto a graphite target for various angles of scattering, measuring the intensity of the scattered x-rays as a function of their wavelength.



Compton discovered that although the incident beam consisted of x-rays of a single wavelength, the scattered x-rays have intensity peaks at two distinct wavelengths. One of them is at the same as the incident one.

1.3.1 The Failure of Electromagnetism

According to electromagnetic theory, the income x-rays should be regarded as electromagnetic waves with a frequency corresponding to the frequency of oscillation of the electric field component of the wave. The free electrons inside the graphite target to start oscillating with the same frequency as the incident wave, and this would cause the oscillating electrons to emit electromagnetic radiation with the same frequency. If the frequency of the emitted radiation is the same as the incident radiation, then so too is the wavelength. Therefore, classical electromagnetism had no prediction about the Compton scattering.

1.3.2 Explaining Observations

The way of Compton explaining these results was to use the photon model of light introduced by Einstein in 1905 through the photoelectric effect. Compton assumed

[4]Notice that there were evidence, not a concrete proof about the nature of light. It was just on the following effect that it became clear what light is.

that the incoming x-ray beam consisted of a stream of photons collided with free electrons in the graphite in a one-to-one collision much like two colliding billiard balls.

According to this view, it is as the individual photons are scattered due to their collision with the free electrons inside the graphite, the incident photon transfers some of its energy to the electron that it collides with, the scattered photon, therefore, will have a lower energy that the incident photon.

By equation (1.1.26) and $c = \nu/\lambda$

$$E = h\nu \iff E = \frac{hc}{\lambda}$$

$$\lambda = \frac{hc}{E} \iff \lambda \propto \frac{1}{E} \tag{1.3.1}$$

Since, $E^2 = p^2c^2 + m^2c^4$, with $m = 0$, $E^2 = p^2c^2 \iff p = \dfrac{E}{c}$

$$\iff E = \frac{hc}{\lambda}$$

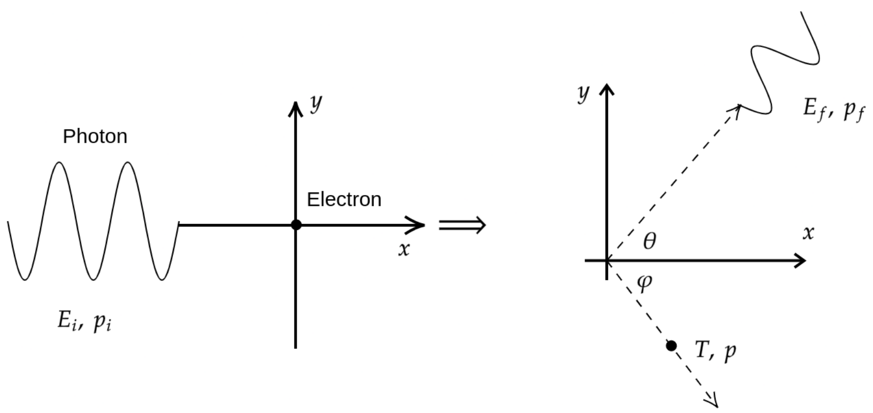
Combining the two above equations,

$$p = \frac{h}{\lambda} \tag{1.3.2}$$

Being the momentum of the photon, which clearly only depends on its wavelength.

Compton then noticed that the wavelength of the scattered radiation was independent of the material contained within the target. So this implied that the scattering process did not involve the whole atom. Therefore, Compton assumed that the scattering was due to collisions between x-ray photons and individual electrons inside the target. There was also the assumption that these electrons would behave exactly as if they ere completely free, provided that the light which hit them, carried larges amounts of energy.

Now, knowing the way through the photons momentum expression, by the conservation of momentum, it is possible to predict the change of wavelength of the scattered photon, likewise the image just bellow.



$$\therefore \begin{cases} p_i = p_f \cos\theta + p\cos\varphi \\ p_f \sin\theta = p\sin\varphi \end{cases} \implies (p_i - p_f\cos\theta)^2 + p_f^2 \sin^2\theta = p^2\cos^2\varphi + p^2\sin^2\varphi$$

$$\iff p^2 = (p_i - p_f\cos\theta)^2 + p_f^2\sin^2\theta$$
$$\iff p^2 = p_i^2 - 2p_ip_f\cos\theta + p_f^2(\sin^2\theta + \cos^2\theta)$$

$$p^2 = p_i^2 + p_f^2 - 2p_ip_f\cos\theta \tag{1.3.3}$$

Next, by the principle of conservation of energy, $E_i + m_ec^2 = E_f + m_ec^2 + T \iff T = E_i - E_f$, since it is the x-ray photon who transfers energy to the electron,

$$T = (p_i - p_f)c$$

Futhermore, by $E = m_ec^2 + T$ and the relativistic equation,

$$(m_ec^2 + T)^2 = p^2c^2 + (mc^2)^2 \iff T^2 + 2Tm_ec^2 = p^2c^2$$

$$p^2 = \frac{T^2}{c^2} + 2Tm_e \tag{1.3.4}$$

Now, putting together equation (1.3.3) and (1.3.4), therefore,

$$p_i^2 + p_f^2 - 2p_ip_f\cos\theta = \frac{T^2}{c^2} + 2Tm_e$$
$$\iff p_i^2 + p_f^2 - 2p_ip_f\cos\theta = \frac{c^2(p_i - p_f)^2}{c^2} + 2c(p_i - p_f)m_e$$
$$\iff m_ec(p_i - p_f) = (1 - \cos\theta)p_ip_f$$

$$\iff \frac{1}{p_f} - \frac{1}{p_i} = \frac{1}{m_ec}(1 - \cos\theta) \iff \frac{h}{p_f} - \frac{h}{p_i} = \frac{h}{m_ec}(1 - \cos\theta)$$

By equation (1.3.2),

$$\lambda' - \lambda = \frac{h}{m_ec}(1 - \cos\theta) \tag{1.3.5}$$

So how does one interpret this result?

The first thing to notice is that the shifted wavelength only depend on the angle of scattering. It does not depend on the wavelength of the incoming radiation. Secondly, since the Compton wavelength is given by,

$$\lambda_c = \frac{h}{m_ec} = 2.43 \cdot 10^{-12} \text{ m} \tag{1.3.6}$$

it sets the scale for the wavelength shift.

The remarkable agreement between theory and experiment signaled success for Compton's photon model approach to x-ray scattering.

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