Lorentz Group From an Undergrad Student

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Chapter 1

A First Sight At The Lorentz Group

1.1 The Lorentz Group

Let there be the set of all matrices 4x4 denominated by $\mathcal{L}$ equipped with matrix multiplication as a operation such that the metric is preserved, i.e, the following expression is satisfied:

$$\Lambda^T \eta \Lambda = \eta \tag{1.1}$$

$\forall \in \mathcal{L}$ such that $\eta = \text{diag}(-1, +1, +1, +1)$

Notice how this set $\mathcal{L}$ is closed since by taking arbitrary $\Lambda_1, \Lambda_2 \in \mathcal{L}$.

$$\implies (\Lambda_1 \Lambda_2)^T \eta \Lambda_1 \Lambda_2 = \eta \iff \Lambda_1^T \Lambda_2^T \eta \Lambda_1 \Lambda_2 = \eta$$

$$\iff \Lambda_2^T \Lambda_1^T \eta \Lambda_1 \Lambda_2 = \eta$$

See that in the above equation we've just obtained $\Lambda_1^T \eta \Lambda_1$ which by our initial restriction $\Lambda_1^T \eta \Lambda_1 = \eta$ which directly implies,

$$\Lambda_2^T \Lambda_1^T \eta \Lambda_1 \Lambda_2 = \eta \iff \Lambda_2^T \eta \Lambda_2 = \eta \quad \forall \Lambda \in \mathcal{L}$$

Looking back to the equation (1.1) and taking its determinant on both sides, we can clearly draw some conclusions.

$$\Lambda^T \eta \Lambda = \eta \iff \det \left(\Lambda^T \eta \Lambda \right) = \det \eta$$

$$\iff \det \Lambda^T \det \eta \det \Lambda = \det \eta \iff \det \Lambda^T \det \Lambda = 1$$

Making the assumption that $\Lambda^T = \Lambda$,

$$\implies (\det \Lambda)^2 = 1$$

$$\therefore \det \Lambda = \pm 1 \tag{1.2}$$

So $\mathcal{L}$ has a group structure, the so called **Lorentz Group**, *but* we shall restrictic ourselves only to the subset of $\mathcal{L}$ such that $\det \Lambda = 1 \ \forall \Lambda \in \mathcal{L}$ that also is equipped of a group structure. This subset with group structure has a name, it is denominades as the $\mathbf{SO(1, 3)}$.

Definition 1.1.1 (Lorentz Group)**.** The Lorentz Group with restriction $\det \Lambda = 1$ is called the *Special Orthogonal Group* of 4x4 matrices.

$$\mathbf{SO(1,3)} = \left\{ \Lambda_{4x4};\ \Lambda^{\mu}_{\nu} \in \mathbb{R},\ \ \Lambda^T \eta \Lambda = \eta,\ \ \det \Lambda = 1 \right\}$$

1.2 Representation Theory and The Lorentz Group

Definition 1.2.1 (Representation)**.** A representation of a group G is a *homomorphism*, i.e, a surjective map, from G to $GL(V)$.

$$\forall g \in G; \;\; g \to D(g) \in GL(V)$$

Properties of a Representation:

Property 1. Multiplication: $D(g_1)D(g_2) = D(g_1 g_2)$

Property 2. Identity Element: $D(e) = \mathbb{1}$

Property 3. Inverse Element: $D(g^{-1}) = \Big[D(g)\Big]^{-1} = D^{-1}(g)$

Property 4. Associativity: $D(g_1)\Big(D(g_2)D(g_3)\Big) = \Big(D(g_1)D(g_2)\Big)D(g_3)$

Definition 1.2.2 (Reducible Representations)**.** A representation D exists if there exists a subspace U, s.t, $U \neq 0$. So for every reducible representation we can write the following:

$$D(g) = \begin{pmatrix} D(g_1) & 0 & 0 & \cdots & 0 \\ 0 & D(g_2) & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \end{pmatrix} \iff D(g) = D(g_1) \oplus D(g_2) \cdots$$

Otherwise, $D(g)$ is a irreducible representation.

Notice how irreducible representations are way more fundamental and therefore hugely interesting in mathematical terms of a physical theory.

One of the most important representations when talking about Lie Groups is the idea of a *exponential map*

Definition 1.2.3 (Exponencial Map)**.** The representation of each of the group elements expressed as a function of parameters $\alpha_i \in \mathbb{R}$, s.t, each parameter is associated with a generator.

$$D(\alpha) = e^{i\alpha_i X_i}$$

X_i are the generators related of the respect Lie Group, which follow a Lie Algebra.

We can further investigate $\mathbf{SO(1,3)}$ by taking its exponential map representation.

$$D(\Lambda) = e^{i\omega} \tag{1.3}$$

Looking back at equation (1.1),

$$\Lambda^T \eta \Lambda = \eta \iff \Lambda^T \eta \Lambda \Lambda^{-1} = \eta \Lambda^{-1} \iff \Lambda^T \eta = \eta \Lambda^{-1} \iff \eta^{-1} \Lambda^T \eta = \eta^{-1} \eta \Lambda^{-1}$$

$$\therefore \eta^{-1} \Lambda^T \eta = \Lambda^{-1} \tag{1.4}$$

Now taking equation (1.4) and applying the exponencial map,

$$\eta^{-1}(e^{i\omega})^T\eta = (e^{i\omega})^{-1} \iff \eta^{-1}e^{i\omega^T}\eta = e^{-i\omega} \tag{1.5}$$

One of the properties of matrices as exponents is:

$$e^{A^{-1}BA} = A^{-1}e^B A \tag{1.6}$$

Therefore applying (1.6) in (1.5) results in

$$\eta^{-1}e^{i\omega^T}\eta = e^{-i\omega} \iff e^{i\eta^{-1}\omega^T\eta} = e^{-i\omega}$$

$$\implies \eta^{-1}\omega^T\eta = -\omega \iff \eta\eta^{-1}\omega^T\eta = -\eta\omega$$

$$\therefore \omega^T\eta = -\eta\omega \tag{1.7}$$

This restriction from above directly implies that the matrix ω necessarily is made of a symmetric and a anti-symmetric part, then, it can be written as

$$\omega = \begin{pmatrix} 0 & a & b & c \\ a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & d & e \\ 0 & -d & 0 & f \\ 0 & -e & -f & 0 \end{pmatrix} \; ; \quad a, b, d, e, f \in \mathbb{R} \tag{1.8}$$

But the above expression still can be simplified in a way more compact way.

$$\omega = aT_a + bT_b + cT_c + dT_d + eT_e + fJ_f \tag{1.9}$$

such that

$$T_a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; T_b = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; T_c = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{1.10}$$

$$T_d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; T_e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; T_f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{1.11}$$

Previously we had the restriction on equation (1.3), since $\det\Lambda = 1$ and $\det\Lambda = \det e^{i\omega} = 1$. However $\det e^{i\omega} = e^{i\,\mathrm{Tr}\,\omega} = 1$ which makes $\mathrm{Tr}\,\omega = 0$. Making use of index notation for a mathematical object with two indices and using the fact that the whole ω matrix is anti-symmetric, then $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$. In a very similar way it is possible to enumerate the matrices J_a, T_b, T_c, T_d, T_e, T_f

$$\therefore \begin{cases} T_a \equiv T_{01} & T_d \equiv T_{12} \\ T_b \equiv T_{02} & T_e \equiv T_{13} \\ T_c \equiv T_{03} & T_f \equiv T_{23} \end{cases} \tag{1.12}$$

A important thing to notice is how we can completely rewrite the ω matrix in terms of repeated indices,

$$\omega = \begin{pmatrix} 0 & \omega^{01} & \omega^{02} & \omega^{03} \\ \omega^{01} & 0 & \omega^{12} & \omega^{13} \\ \omega^{02} & -\omega^{12} & 0 & \omega^{23} \\ \omega^{03} & -\omega^{13} & -\omega^{23} & 0 \end{pmatrix}$$

When performing the sum in index notation of equation (1.9) a factor of 2 appears on the right-hand of the equation, so get rid of it, we divide by a factor of 2 on both sides of the equation. With all of that in mind we can finally write in compact notation the ω matrix.

$$\omega = \frac{1}{2}\omega^{\alpha\beta}T_{\alpha\beta} \tag{1.13}$$

$$\implies D(\Lambda) = e^{\frac{i}{2}\omega^{\alpha\beta}T_{\alpha\beta}} \tag{1.14}$$

See that the ω matrix is part of a linear vector space, which allow us as already verified on equation (1.9) that every ω can be written as a linear combination of the $T_{\alpha\beta}$ matrices. By the definition 1.2.3, $T_{\alpha\beta}$ are the generators of the group, since generators follow a *Lie Algebra*, there comes the related *Lie Algebra* of the $\mathbf{SO(1,3)}$ group.

$$\therefore \mathfrak{so}(1,3) = \left\{ \omega = \frac{i}{2}\omega^{\alpha\beta}T_{\alpha\beta} \middle| \omega^{\alpha\beta} = -\omega^{\beta\alpha} \right\} \tag{1.15}$$